Generic linear perturbations

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Main references

1. J. N. Mather, *Generic projections*, Ann. of Math., (2) **98** (1973), 226–245.
2. S. Ichiki, *Generic linear perturbations*, arXiv:1607.03220v2 (preprint).
3. S. Ichiki, *Composing generic linearly perturbed mappings and immersions/injections*, arXiv:1612.01100 (preprint).

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(Main references)

[1] J. N. Mather, *Generic projections*, Ann. of Math., (2) **98** (1973), 226–245.

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All mappings and manifolds belong to class C^∞ in this talk.

- N, P : manifold
- $C^\infty(N, P)$: the set of C^∞ mappings of N into P
(the topology on $C^\infty(N, P)$: Whitney C^∞ topology)
- $g : N \to P$ is *$\mathcal{A}$-equivalent* to $h : N \to P$
$\overset{def}{\Leftrightarrow}$ $\exists$ diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$
s. t. $g = \Psi \circ h \circ \Phi^{-1}$.
- $g : N \to P$: *stable*
$\overset{def}{\Leftrightarrow}$ the $\mathcal{A}$-equivalence class of g is open in $C^\infty(N, P)$.

The following problem was proposed by René F. Thom.

Structural stability problem

Are the stable mappings of N into P dense in $C^\infty(N, P)$?

René F. Thom (1923–2002)

写真：https://ja.wikipedia.org/wiki/ルネ・トム より引用

- The celebrated series by John N. Mather
  " Stability of C^∞ mappings I, II, III, IV, IV, VI, (1968–1971)"
  are essential for the stability of C^∞ mappings.
- In the sixth paper "Stability of C^∞ mappings VI: The nice dimensions", Mather stated the following answer (next page) to the structural stability problem.

John N. Mather (1942–2017)

写真：https://www.cwi.nl/news/2014/mathematician-john-norman-mather-receives-brouwer-medal-2014 より引用

The following result by Mather is well known.

Theorem 1 (Mather)

- *N : compact manifold of dimension n*
- *P : manifold of dimension p*

Then, stable mappings in $C^\infty(N, P)$ are dense if and only if the pair (n, p) satisfies one of the following conditions.

(1) $n < \frac{6}{7}p + \frac{8}{7}$ *and* $p - n \geq 4$

(2) $n < \frac{6}{7}p + \frac{9}{7}$ *and* $3 \geq p - n \geq 0$

(3) $p < 8$ *and* $p - n = -1$

(4) $p < 6$ *and* $p - n = -2$

(5) $p < 7$ *and* $p - n \leq -3$

- (n, p) : *nice dimension*
  $\overset{def}{\Leftrightarrow}$ the stable mappings in $C^\infty(N, P)$ are dense.

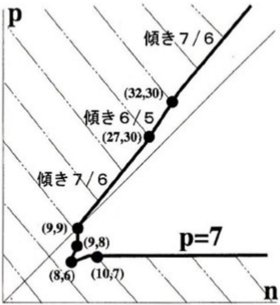

Figure: nice dimensions (striped area)

図：特異点と分岐, 特異点の数理２（西村尚史, 福田拓生）共立出版, 2002. より引用

After the celebrated series "Stability of C^∞ mappings I, II, III, IV, IV, VI, (1968–1971)", Mather also showed striking results in the paper "Generic projections (1973)".

- $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: set consisting of linear mappings $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$
- We have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$.
- By the symbol Σ, we denote a measure zero set of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ $(= (\mathbb{R}^m)^\ell)$.

As a striking result of "Generic projections", the following is well known.

Theorem 2 (Mather)

- N : *compact manifold of dimension* n
- $f : N \to \mathbb{R}^m$: *embedding*
- $m > \ell$
- (n, ℓ) : *nice dimension*

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ *is stable.*

Motivation

- $F : U \to \mathbb{R}^\ell$: mapping ($U \subset \mathbb{R}^m$: open)
- $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$: linear mapping
- Set

$$F_\pi = F + \pi.$$

In "Generic projections", for a given embedding $f : N \to \mathbb{R}^m$, a composition $\pi \circ f : N \to \mathbb{R}^\ell$ ($m > \ell$) is investigated from the viewpoint of stability.
On the other hand, in "today's talk", for a given embedding $f : N \to U$, properties of a composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ are introduced. (Finally, as some related results, for a given immersion $f : N \to U$, properties of a composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ are also introduced.)

Before stating the main theorems of "Generic projections" and my talk, some definitions are prepared.

- N, P : manifold
- W : submanifold of P

Definition 1

- $f : N \to P$ is *transverse* to W ($f \pitchfork W$)
  $\overset{def}{\Leftrightarrow}$ *for any* $q \in N$, *either one of the following two holds:*
  1. $f(q) \notin W$.
  2. $f(q) \in W$ *and* $df_q(T_qN) + T_{f(q)}W = T_{f(q)}P$.

- N, P : manifold
- $J^r(N, P)$: the space of r-jets of mappings of N into P
- For a given mapping $g : N \to P$, $j^r g : N \to J^r(N, P)$ is defined by $q \mapsto j^r g(q)$.
- $N^{(s)} = \{q = (q_1, \ldots, q_s) \in N^s \mid q_i \neq q_j (1 \leq i < j \leq s)\}$
- ${}_sJ^r(N, P)$ is defined as follows: $\{(j^r g_1(q_1), \ldots, j^r g_s(q_s)) \in J^r(N, P)^s \mid q \in N^{(s)}\}$.
- For $g : N \to P$, the mapping ${}_sj^r g : N^{(s)} \to {}_sJ^r(N, P)$ is defined by $(q_1, \ldots, q_s) \mapsto (j^r g(q_1), \ldots, j^r g(q_s))$.

- π : a *partition* of $\{1, \ldots, s\}$
  $\overset{def}{\Leftrightarrow}$
  1. $\exists A_1, \ldots, A_m \subset \{1, \ldots, s\}$ s. t. $\pi = \{A_1, \ldots, A_m\}$
  2. $\bigcup_{i=1}^{m} A_i = \{1, \ldots, s\}$
  3. $\forall i \ (1 \leq i \leq m)$, $A_i \neq \emptyset$
  4. $\forall i, j \ (i \neq j)$, $A_i \cap A_j = \emptyset$
- Following Mather, we define P^π as follows:
  $P^\pi = \{(y_1, \ldots, y_s) \in P^s \mid y_i = y_j \iff i$ and j are in the same member of the partition π $\}$
- W $(\subset {}_sJ^r(N, P))$: *invariant*
  $\overset{def}{\Leftrightarrow}$ ${}_sj^r g(q) \in W \Rightarrow {}_sj^r(H \circ g \circ h^{-1})(q') \in W$,
  for any diffeomorphisms $H : P \to P$ and $h : N \to N$
  $(q = (q_1, \ldots, q_s), q' = (h(q_1), \ldots, h(q_s)))$.

In order to introduce the main theorem of "Generic projections", it is necessary to prepare the notion "modular submanifold " defined by Mather.

- $q = (q_1, \dots, q_s) \in N^{(s)}$
- $g : N \to P$
- $q' = (g(q_1), \dots, g(q_s))$

- $J^r(N,P)_{q_i}$: the fiber of $J^r(N,P)$ over q_i $(1 \leq i \leq s)$
- ${}_sJ^r(N,P)_q$ (resp., ${}_sJ^r(N,P)_{q,q'}$) : the fiber of ${}_sJ^r(N,P)$ over q (resp., over (q,q')).
- $J^r(N)_q = \bigoplus_{i=1}^s J^r(N,\mathbb{R})_{q_i}$: $\mathbb{R}$-algebra
- $\mathfrak{m}_q = \bigoplus_{i=1}^s \mathfrak{m}_{q_i}$
  $(\mathfrak{m}_{q_i} = \{j^r h_i(q_i) \in J^r(N,\mathbb{R})_{q_i} \mid h_i : (N,q_i) \to (\mathbb{R},0)\})$

Preliminaries

- $q = (q_1, \ldots, q_s) \in N^{(s)}$
- $g : N \to P$
- $q' = (g(q_1), \ldots, g(q_s))$

- TP : tangent bundle of P
- $g^*TP = \bigcup_{x \in N} T_{g(x)}P$
- $J^r(g^*TP)_{q_i} = \{j^r\xi(q_i) \in J^r(N, g^*TP) \mid \xi : (N, q_i) \to g^*TP, \pi_g \circ \xi = id_{(N,q_i)}\}$,
  where
  - $\pi_g : g^*TP \to N$ $(\pi_g(v_{g(x)}) = x)$
  - $id_{(N,q_i)} : (N, q_i) \to (N, q_i)$: identity
- $J^r(g^*TP)_q = \bigoplus_{i=1}^{s} J^r(g^*TP)_{q_i}$: $J^r(N)_q$-module
- $\mathfrak{m}_q J^r(g^*TP)_q$: set consisting of finite sums of the product of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$

Then, one can show the following canonical identification of $\mathbb{R}$ vector spaces.

$$T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_q J^r(g^*TP)_q, \qquad (*)$$

where $q \in N^{(s)}$, $z = {}_sj^r g(q)$ and $q' = (g(q_1), \ldots, g(q_s))$.

- W : submanifold of ${}_sJ^r(N,P)$
- Choose $q = (q_1, \ldots, q_s) \in N^{(s)}$ and $g : N \to P$ satisfying ${}_sj^r g(q) \in W$
- Set $z = {}_sj^r g(q)$ and $q' = (g(q_1), \ldots, g(q_s))$.
- $W_{q,q'}$: the fiber of W over (q, q').

Then, under the identification $(*)$, $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. We denote this vector subspace by $E(g, q, W)$.

Definition 3 (Mather)

The submanifold W is called *modular* if conditions (α) and (β) below are satisfied:

(α) W is an invariant submanifold of ${}_sJ^r(N, P)$, and $\exists$ partition π of $\{1, \ldots, s\}$ s. t. $\Pi(W) \subset P^\pi$, where $\Pi : {}_sJ^r(N, P) \to P^s$ $(\Pi({}_sj^r g(q)) = (g(q_1), \ldots, g(q_s)))$.

(β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ such that ${}_sj^r g(q) \in W$, the subspace $E(g, q, W)$ is a $J^r(N)_q$-submodule.

The following is the main theorem of "Generic projections".

Theorem 4 (Mather)

- N : *manifold of dimension* n
- $f : N \to \mathbb{R}^m$: *embedding*
- W : *modular submanifold of* ${}_sJ^r(N, \mathbb{R}^\ell)$
- $m > \ell$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, ${}_sj^r(\pi \circ f) \pitchfork W$.

As one of the applications of the main theorem of "Generic projections", the following is well known.

Theorem 5 (Mather)

- N : *compact manifold of dimension* n
- $f : N \to \mathbb{R}^m$: *embedding*
- $m > \ell$
- (n, ℓ) : *nice dimension*

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ *is stable.*

The main theorem of my talk is the following.

Theorem 6 (I)

- N : *manifold of dimension* n
- $f : N \to U$: *embedding* ($U \subset \mathbb{R}^m$: *open*)
- $F : U \to \mathbb{R}^\ell$
- W : *modular submanifold of* ${}_sJ^r(N, \mathbb{R}^\ell)$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, ${}_sj^r(F_\pi \circ f) \pitchfork W$.

Remark that in the case of $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, Theorem 6 is the main theorem of "Generic projections".

Corollary 7 (I)

- *N : compact manifold of dimension n*
- *$f : N \to U$: embedding ($U \subset \mathbb{R}^m$: open)*
- $F : U \to \mathbb{R}^\ell$
- (n, ℓ) : *nice dimension*

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$ *is stable.*

Remark that in the case of $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, Corollary 7 is the previous Mather's theorem (Theorem 5).

Remark on the main theorem of this talk

We explain the advantage that the domain of the mapping F is an arbitrary open set.
Suppose that

- $U = \mathbb{R}$.
- $F : \mathbb{R} \to \mathbb{R}$ $(x \mapsto |x|)$

Since F is not differentiable at $x = 0$, we can not apply the main theorem to the mapping $F : \mathbb{R} \to \mathbb{R}$.
On the other hand, if $U = \mathbb{R} - \{0\}$, then the theorem can be applied to the restriction $F|_U$.

In order to give the sketch of the proof of the main theorem of my talk, we recall the theorem as follows.

Theorem 6 (I)

- N : manifold of dimension n
- $f : N \to U$: embedding ($U \subset \mathbb{R}^m$: open)
- $F : U \to \mathbb{R}^\ell$
- W : modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: measure zero set
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, ${}_sj^r(F_\pi \circ f) \pitchfork W$.

For the proof, the Mather's theorem of "Generic projections" is essential as a lemma. The proof is technical, but simple.
(Proof)

- $(\alpha_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$: representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$
- Set $F_\alpha = F_\pi$, and we have

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j}x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j}x_j \right),$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$,
$F = (F_1, \ldots, F_\ell)$ and $x = (x_1, \ldots, x_m)$.

- For the given embedding $f : N \to U$, a mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is as follows:

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, \dots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right),$$

where $f = (f_1, \dots, f_m)$.

By the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove the main theorem, it is sufficient to show the following.

Our aim

$\exists \Sigma \subset (\mathbb{R}^m)^\ell$: measure zero set
s. t. $\forall \alpha \in (\mathbb{R}^m)^\ell - \Sigma$,
${}_sj^r(F_\alpha \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to the given modular submanifold W.

Let $H_\Lambda : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the linear isomorphism defined by

$$H_\Lambda(X_1, \ldots, X_\ell) = (X_1, \ldots, X_\ell)\Lambda,$$

where $\Lambda = (\lambda_{ij})_{1\le i\le \ell, 1\le j\le \ell}$ is an $\ell \times \ell$ regular matrix.
The composition of H_Λ and $F_\alpha \circ f$ is as follows (in the next page):

$$
\begin{aligned}
& H_\Lambda \circ F_\alpha \circ f \\
= & \left(\sum_{k=1}^{\ell} \left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j \right) \lambda_{k1}, \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j \right) \lambda_{k\ell} \right) \\
= & \left(\sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k1} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k1} \alpha_{kj} \right) f_j, \right. \\
& \left. \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k\ell} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k\ell} \alpha_{kj} \right) f_j \right).
\end{aligned}
$$

Set $GL(\ell) = \{B \mid B : \ell \times \ell \text{ matrix}, \det B \neq 0\}$.
Let $\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell$ be the mapping as follows:

$$\begin{aligned}&\varphi(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{\ell m})\\ &= \Bigg(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k1}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1},\\ &\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k2}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{km}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km}\Bigg).\end{aligned}$$

In fact, φ is a C^∞ diffeomorphism.

- Structure of $\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell$

$$
\begin{aligned}
& \varphi(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{\ell m}) \\
= \; & \Bigg(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k1}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1}, \\
& \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k2}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{km}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km} \Bigg).
\end{aligned}
$$

$$
\begin{aligned}
& H_\Lambda \circ F_\alpha \circ f \\
= \; & \left(\sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k1} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{kj} \right) f_j, \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k\ell} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{kj} \right) f_j \right).
\end{aligned}
$$

Next, let $\widetilde{f} : U \to \mathbb{R}^{m+\ell}$ be the mapping as follows:

$$\widetilde{f}(x_1, \ldots, x_m) = (F_1(x_1, \ldots, x_m), \ldots, F_\ell(x_1, \ldots, x_m), x_1, \ldots, x_m).$$

$\Rightarrow \widetilde{f}$: embedding.

Since $f : N \to U$ is an embedding, $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$ is also an embedding:

$$\widetilde{f} \circ f = (F_1 \circ f, \ldots, F_\ell \circ f, f_1, \ldots, f_m).$$

Sketch of the proof of the main theorem

- $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$: embedding
- W : modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$

By applying the main theorem of "Generic projections", it follows that
$\exists \Sigma \subset \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell)$: measure zero set
s. t. $\forall \Pi \in \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell) - \Sigma$,
${}_sj^r(\Pi \circ \widetilde{f} \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W.

By the natural identification $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^{\ell}) = \mathbb{R}^{\ell(m+\ell)}$,
we identify the target space $GL(\ell) \times (\mathbb{R}^m)^{\ell}$ of the mapping
$\varphi : GL(\ell) \times (\mathbb{R}^m)^{\ell} \to GL(\ell) \times (\mathbb{R}^m)^{\ell}$
with an open submanifold of $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^{\ell})$.
Then, we have

- $(GL(\ell) \times (\mathbb{R}^m)^{\ell}) \cap \Sigma$: measure zero set of $GL(\ell) \times (\mathbb{R}^m)^{\ell}$
- $\varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^{\ell}) \cap \Sigma)$: measure zero set of $GL(\ell) \times (\mathbb{R}^m)^{\ell}$

For any $(\Lambda, \alpha) \in GL(\ell) \times (\mathbb{R}^m)^\ell$, let $\Pi_{(\Lambda,\alpha)} : \mathbb{R}^{m+\ell} \to \mathbb{R}^\ell$ be the linear mapping defined by $\varphi(\Lambda, \alpha)$ as follows:

$$\Pi_{(\Lambda,\alpha)}(X_1, \ldots, X_{m+\ell}) = (X_1, \ldots, X_{m+\ell}) \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1\ell} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell 1} & \cdots & \lambda_{\ell\ell} \\ \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1} & \cdots & \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km} & \cdots & \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km} \end{pmatrix}.$$

Then, we have the following:

$$
\begin{aligned}
& \Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f \\
= \; & \left(\sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k1} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k1} \alpha_{kj} \right) f_j, \ldots, \right. \\
& \left. \sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k\ell} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k\ell} \alpha_{kj} \right) f_j \right) \\
= \; & H_{\Lambda} \circ F_{\alpha} \circ f.
\end{aligned}
$$

Therefore, for any $(\Lambda, \alpha) \in GL(\ell) \times (\mathbb{R}^m)^{\ell} - \varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^{\ell}) \cap \Sigma)$, it follows that ${}_s j^r(\Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f)$ $(= {}_s j^r(H_{\Lambda} \circ F_{\alpha} \circ f))$ is transverse to W.

Set $\widetilde{\Sigma} = \{\alpha \in (\mathbb{R}^m)^\ell \mid {}_sj^r(F_\alpha \circ f) \text{ is not transverse to } W\}$.
Suppose $\widetilde{\Sigma}$ is not measure zero set of $(\mathbb{R}^m)^\ell$.
$\Rightarrow$

- $GL(\ell) \times \widetilde{\Sigma}$ is not a measure zero set of $GL(\ell) \times (\mathbb{R}^m)^\ell$.
- $\forall(\Lambda, \alpha) \in GL(\ell) \times \widetilde{\Sigma}$, ${}_sj^r(H_\Lambda \circ F_\alpha \circ f)$ is not transverse to W.

This contradicts the assumption $\varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$ is measure zero.

□

In the main theorem of this talk, the given mapping $f : N \to U$ is an embedding.
On the other hand, finally, I'd like to introduce some related results in the case that the given mapping $f : N \to U$ is an immersion. The reference is the following:

[3] S. Ichiki, *Composing generic linearly perturbed mappings and immersions/injections*, arXiv:1612.01100 (preprint).

- $J^1(n,\ell) = \{j^1 g(0) \mid g : (\mathbb{R}^n, 0) \to (\mathbb{R}^\ell, 0)\}$
- $X \subset J^1(n,\ell)$: *invariant*
  $\overset{def}{\Leftrightarrow} j^1 g(0) \in X \Rightarrow j^1(H \circ g \circ h^{-1})(0) \in X$,
  for any two germs of diffeomorphisms
  $H : (\mathbb{R}^\ell, 0) \to (\mathbb{R}^\ell, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

Theorem 8 (I)

- N : *manifold of dimension* n
- $f : N \to U$: *immersion* ($U \subset \mathbb{R}^m$: *open*)
- $F : U \to \mathbb{R}^\ell$
- $X \subset J^1(n, \ell)$: *invariant submanifold*

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ *is transverse to* $X(N, \mathbb{R}^\ell)$,
where $X(N, \mathbb{R}^\ell)$ *is a sub-fiber bundle of* $J^1(N, \mathbb{R}^\ell)$ *with the given fiber* X.

The following well known result lies at the heart of most applications of transversality.

Lemma 9 (the fundamental transversality lemma)

- X, A, Y : *manifold*
- Z : *submanifold of* Y
- $F : X \times A \to Y$: *mapping*
- $F \pitchfork Z$

$\Rightarrow \exists \Sigma$: *measure zero set of* A
s. t. $\forall a \in A - \Sigma$, $F_a \pitchfork Z$,
where $F_a : X \to Y$ $(x \mapsto F(x, a))$.

Corollary 10 (I)

- N : *manifold of dimension* n
- $f : N \to U$: *immersion* ($U \subset \mathbb{R}^m$: *open*)
- $F : U \to \mathbb{R}$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R})$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$,
any singular point of $F_\pi \circ f : N \to \mathbb{R}$ *is nondegenerate.*

- N : manifold of dimension n
- $g : N \to \mathbb{R}^{2n-1}$ $(n \geq 2)$
- $q \in N$: singular point of Whitney umbrella of g
  $\overset{def}{\Leftrightarrow}$ $\exists H : (\mathbb{R}^{2n-1}, g(q)) \to (\mathbb{R}^{2n-1}, 0)$ and
  $\exists h : (N, q) \to (\mathbb{R}^n, 0)$: germs of diffeomorphisms
  s. t.

$$\begin{aligned} & H \circ g \circ h^{-1}(x_1, \ldots, x_n) \\ = \ & (x_1^2, x_1x_2, \ldots, x_1x_n, x_2, \ldots, x_n). \end{aligned}$$

Corollary 11 (I)

- N : *manifold of dimension* n $(n \geq 2)$
- $f : N \to U$: *immersion* $(U \subset \mathbb{R}^m$: *open*$)$
- $F : U \to \mathbb{R}^{2n-1}$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$,
any singular point of $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$
is a singular point of Whitney umbrella.

Corollary 12 (I)

- N : *manifold of dimension* n
- $f : N \to U$: *immersion* ($U \subset \mathbb{R}^m$: *open*)
- $F : U \to \mathbb{R}^\ell$
- $\ell \geq 2n$

$\Rightarrow \exists \Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *measure zero set*
s. t. $\forall \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \to \mathbb{R}^\ell$: *immersion*.

Summary of this talk

- $F : U \to \mathbb{R}^\ell$ ($U \subset \mathbb{R}^m$: open)
- $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$: linear mapping
- $F_\pi = F + \pi$

- In the celebrated paper "Generic projections" by Mather, for a given embedding $f : N \to \mathbb{R}^m$, a composition $\pi \circ f : N \to \mathbb{R}^\ell$ is investigated (the case of $F = 0$).
- On the other hand, in this talk, an improvement of the main theorem of "Generic projections" is given. Namely, for a given embedding $f : N \to U$, a composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is investigated.
- As some related results, for a given immersion $f : N \to U$, some properties of a composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ are also introduced.

Thank you for your attention.