A_∞代数の幾何学への応用のいくつかについて

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（少し修正版）

Def. [A_∞-algebra (Stasheff'63)]

$(A, \mathfrak{m} := \{m_k\}_{k\geq 1})$ *is an* A_∞**-algebra** $\Leftrightarrow$

$A = \oplus_{r\in\mathbb{Z}} A^r$ *: $\mathbb{Z}$-graded vector space,*

$\mathfrak{m} := \{m_n : A^{\otimes n} \to A\}_{n\geq 1}$ *: linear maps of degree* $|m_n| = (2-n)$

s.t.

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm\, m_k(a_1, \cdots, a_j, m_l(a_{j+1}, \cdots, a_{j+l}), a_{j+l+1}, \cdots, a_n)\ ,$$

for $n = 1, 2, \ldots,$ *where* $a_i \in A^{|a_i|}$, $i = 1, \ldots, n$.

$|m_n| = (2-n)$ *implies*

$$|m_n(a_1, ..., a_n)| = (2-n) + |a_1| + \cdots + |a_n|.$$

The A_∞-relations for $n = 1, 2, 3$:

for $m_1 = d$, $m_2 = \cdot$, $\quad x, y, z \in V$:

$$i) \quad d^2 = 0 \ ,$$

$$ii) \quad d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) \ ,$$

$$iii) \quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z).$$

$i) \Leftrightarrow (A, d)$ forms a complex.

$ii) \Leftrightarrow$ Leibniz rule of d w.r.t. the product $\cdot$.

$iii)$ $\cdot$ is associative **up to homotopy**.

In particular, if $m_3 = 0$, the product $\cdot$ is strictly associative.

An A_∞-algebra $(A, \mathfrak{m})$ with $m_3 = m_4 = \cdots = 0$ is a **DG algebra**.

An A_∞-algebra $(A, \mathfrak{m})$ with $m_1 = 0$ is called **minimal**.

An Example of minimal A_∞-algebra

A generated by $e^0 = id, e^2, e^5$

nontrivial A_∞-product:

$$m_2(e^0, e^2) = m_2(e^2, e^0) = e^2, \qquad m_3(e^2, e^2, e^2) = e^5$$

Def. *Given two* A_∞*-algebras* $(A, \mathfrak{m})$ *and* $(A', \mathfrak{m}')$*, an* A_∞-**morphism** $\mathfrak{f} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *is a collection of degree* $(1-k)$ *multilinear maps* $\mathfrak{f} := \{f_k : A^{\otimes k} \to A'\}_{k \geq 1}$ *s.t.*

$$\sum_{i \geq 1} \sum_{k_1 + \cdots + k_n = n} \pm m'_i (f_{k_1} \otimes \cdots \otimes f_{k_i})(a_1, ..., a_n)$$

$$= \sum_{\substack{i+1+j=k \\ i+l+j=n}} \pm f_k (\mathbf{1}^{\otimes i} \otimes m_l \otimes \mathbf{1}^{\otimes j})(a_1, ..., a_n)$$

for $n = 1, 2, ...$

Note: For $n = 1$: $m'_1 f_1 = f_1 m_1 \Leftrightarrow$

$f_1 : (A, m_1) \to (A', m'_1)$ forms a chain map.

Def. $\mathfrak{f} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *is called an* A_∞-**(quasi)-isomorphism** *iff* $f_1 : (A, m_1) \to (A', m'_1)$ *is a (quasi)-isomorphism.*

Note that A_∞-quasi-isomorphisms define an equivalence relation.

cf. DGA equivalence

a DGA $\to \leftarrow \to \ \cdots \ \to \leftarrow$ another DGA

chains of DG quasi-isomorphisms

Two important theorems:

Minimal model theorem (Kadeishvili'83)

For any A_∞-algebra $(A, \mathfrak{m})$, there exists an A_∞-algebra $(H(A), \mathfrak{m}')$ and an A_∞-quasi-isomorphism $(H(A), \mathfrak{m}') \to (A, \mathfrak{m})$.

Note that $\mathfrak{m}' = \{m'_1 = 0, m'_2, m'_3, \dots\}$. Such an A_∞-algebra $(H(A), \mathfrak{m}')$ is called a **minimal model** of $(A, \mathfrak{m})$.

$\star$ Minimal models of $(A, \mathfrak{m})$ are unique up to A_∞-isomorphisms on $H(A)$.

- $H(A, \mathfrak{m}) := (H(A), m'_2)$ forms a $\mathbb{Z}$-graded algebra.

More generally ...

Homological perturbation theory (1986∼)

(Gugenheim, Lambe, Stasheff, Kadeishvili, Huebschmann,...)

For an A_∞-algebra $(A, \mathfrak{m})$ and

$$(V, d) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (A, m_1) \ , \qquad h : A^r \to A^{r-1}$$

which satisfy

$$m_1 h + h m_1 = id_A - \iota \circ \pi, \qquad \pi \circ \iota = id_V$$

there exists an A_∞-algebra $(V, \mathfrak{m}')$ with $m'_1 = d$ and ι, π lift to A_∞-quasi-isomorphisms.

(∃ an explicit construction using "Feynman graphs".)

A_∞ **Yoneda's lemma (cf. Fukaya'01)**

Any unital A_∞-algebra $(A, \mathfrak{m})$ is A_∞-quasi-isomorphic to a unital DG-algebra.

$$\mathfrak{f} : (A, \mathfrak{m}) \to \text{a DG-algebra}$$

Note: Two DGAs are DGA equivalent $\Leftrightarrow$ Two DGAs are A_∞-quasi-isomorphic to each other.

An application: rational homotopy theory for 1-connected spaces

1-conn. rational homotopy types

$\overset{Quillen'69,\ Sullivan'79}{\Longleftrightarrow}$ DGA equivalence classes of cDGAs

(c = commutative)

$\overset{Kedeishvili'93}{\Longleftrightarrow}$ C_∞-isomorphism classes

of 1-conn. minimal C_∞-algebras

(C_∞ = commutative A_∞-algebra)

Problem: For a fixed cohomology algebra of 1-connected type,

classify its rational homotopy types !

Note: $m_n = 0$ for large n

for minimal A_∞-algebras of 1-connected type.

- A: generated by e^0, e^2, e^5 (cf. $S^2 \vee S^5$)

$$m_3(e^2, e^2, e^2) = a \cdot e^5$$

The resulting minimal A_∞-algebras A_a and A_b are A_∞-isomorphic to each other iff $a = b = 0$ or $a \neq 0 \neq b$.

however, A_a, $a \neq 0$ do not form C_∞-algebras.

- Next example:

A: generated by e^0, e^2, e^3, e^6 (cf. $S^2 \vee S^3 \vee S^6$)

Possible higher A_∞-products:

$$m_3(e^3, e^2, e^2) = a \cdot e^6,$$

$$m_3(e^2, e^3, e^2) = b \cdot e^6,$$

$$m_3(e^2, e^2, e^3) = c \cdot e^6,$$

and

$$m_4(e^2, e^2, e^2, e^2) = d \cdot e^6.$$

C_∞-condition – vanishing on shuffle products in $T(A[1])$

For m_3: we obtain $b = 0$, $c = -a$ since

$$0 = m_3(e^2 \cdot e^2 ⧢ e^3) = m_3(e^2, e^2, e^3) + m_3(e^2, e^3, e^2) + m_3(e^3, e^2, e^2)$$

$$0 = m_3(e^2 \cdot e^3 ⧢ e^2) = m_3(e^2, e^3, e^2) + m_3(e^2, e^2, e^3) - m_3(e^2, e^2, e^3)$$

$$0 = m_3(e^3 \cdot e^2 ⧢ e^2) = m_3(e^3, e^2, e^2) - m_3(e^3, e^2, e^2) - m_3(e^2, e^3, e^2).$$

For m_4: $d = 0$ since

$$0 = m_4(e^2 \cdot e^2 ⧢ e^2 \cdot e^2)$$

$$= (+(+1 - 1 + 1) - (-1 + 1) + (1))m_4(e^2, e^2, e^2, e^2)$$

Thus,

Two rational homotopy types whose cohomology algebras are that of $S^2 \vee S^3 \vee S^6$.

($A(a=-c\neq 0, b=d=0)$ and $A(a=b=c=d=0)$)

- In case $S^{i_1} \vee S^{i_2} \vee \cdots \vee S^{i_n}$, $1 < i_1 < \cdots < i_n$?

$\cdots$ classified in cases $n \leq 7$ by Jyounouchi'16.

only finitely many rational homotopy types in these cases.

The categorical version:

Def. [Fukaya'93] *An* A_∞-**category** $\mathcal{C}$ $\Leftrightarrow$

◦ $\mathrm{O}b(\mathcal{C}) = \{a, b, \cdots\}$ *: a collection of objects*

◦ $\mathcal{C}(a, b) := \oplus_{r\in\mathbb{Z}}\mathcal{C}^r(a, b)$*:* $\mathbb{Z}$*-graded vector space (of morphisms)*

◦ *a collection of multilinear maps*

$$m_n : \mathcal{C}(a_1, a_2) \otimes \cdots \otimes \mathcal{C}(a_n, a_{n+1}) \to \mathcal{C}(a_1, a_{n+1})$$

of degree $(2 - n)$ *defining an* A_∞*-structure.*

In particular, $\mathcal{C}$ *with* $m_3 = m_4 = \cdots = 0$ *is called a* **DG-category**.

- A_∞-functor — analog of A_∞-morphism.
- A_∞-quasi-isomorphism.

  – category version of A_∞-quasi-isomorphism.

- A_∞-equivalence functor

  (defined for A_∞-categories with identity morphisms.)

  – a generalization of an A_∞-quasi-isomorphism so that it defines an equivalence functor on the cohomology categories.

There exists a way to construct a triangulated category $Tr(\mathcal{C})$ from an A_∞-category $\mathcal{C}$:

(Bondal-Kapranov'91(DG), Kontsevich'94 (A_∞))

$$\left\{ \begin{array}{c} A_\infty\text{-category} \\ \mathcal{C} \end{array} \right\} \to \left\{ \begin{array}{c} A_\infty\text{-category} \\ Pre\text{-}Tr(\mathcal{C}) \end{array} \right\}$$

$$\downarrow H^0$$

$$\left\{ \begin{array}{c} \text{tri. category} \\ Tr(\mathcal{C}) \end{array} \right\}$$

It is known that $Tr(\mathcal{C}) \simeq Tr(\mathcal{C}')$ if $\mathcal{C} \simeq \mathcal{C}'$.   (cf. Seidel'08)

This construction is applied to ...

Homological mirror symmetry conjecture (Kontsevich'94) :

If a symplectic manifold M and a complex manifold $\check{M}$ are mirror dual to each other, then

$$Tr(Fuk(M)) \simeq D^b(coh(\check{M}))$$

holds, where $Fuk(M)$ is the Fukaya (A_∞-)category on M.

How to understand this duality ?

Kontsevich-Soibelman'00$\sim$

Usually, the complex side $D^b(coh(\check{M}))$ has a natural DG-structure so that

$$D^b(coh(\check{M})) \simeq Tr(DG(hol(\check{M})))$$

$DG(hol(\check{M}))$: DG category of holomorphic vector bundles on $\check{M}$

cf.

DGA $(\Omega(\check{M}), d, \wedge)$ of differential forms on $\check{M}$

⋮

DGA $(\Omega^{0,*}(\check{M}), \bar{\partial}, \wedge)$ of anti-holomorphic differential forms on $\check{M}$

⋮

DG category $DG(hol(\check{M}))$ where each holomorphic vector bundle has a holomorphic structure $D := \bar{\partial} + A^{0,1}$, $D^2 = 0$.

Hope we can take full subcategories

$$\mathcal{C} \subset Fuk(M), \qquad \mathcal{C}' \subset DG(hol(\check{M}))$$

such that $Tr(\mathcal{C}) \simeq Tr(Fuk(M)),\ Tr(\mathcal{C}') \simeq Tr(DG(hol(\check{M})))$,

and $\mathcal{C} \simeq \mathcal{C}'$ as A_∞-categories. $\quad \Rightarrow \quad$ This implies $Tr(\mathcal{C}') \simeq Tr(\mathcal{C})$.

$\star$ Forthermore, the A_∞-equivalence $\mathfrak{f} : \mathcal{C} \to \mathcal{C}'$ is obtained

by the homological perturbation theory !

Outline to obtain $\mathfrak{f} : \mathcal{C} \to \mathcal{C}'$:

B: n-dim. mfd (equipped with tropical affine, Hessian structures!)

$\cdots \to T^*B$: symplectic manifold

$\cdots \to M := T^*B/\mathbb{Z}^n$: symplectic torus fibration

A_∞**-category** $M(B)$ **of Morse homotopy on** B:

$Ob(M(B)) = C^\infty(B)$,

For $f, g \in Ob(M(B))$, $Hom(f,g)$ is the Morse complex of $f - g$.

• Fukaya, Oh'93,'97: $M(B)$ is equivalent to the full subcategory of $Fuk(T^*B)$ consisting of Lagrangian sections $graph(df)$.

- $M(B)$ is A_∞-quasi-isomorphic to **a DG category** $DG(B)$

  via the homological perturbation theory

  where $Ob(DG(B)) = Ob(M(B))$,

  $$Hom_{DG(B)}(f,g) = \Omega(B), \qquad D = d + df\wedge$$

- Extend these stories to torus fibrations

- The DG structure in $DG(B)$ corresponds to that in $DG(hol(\check{M}))$

  where $\check{M} := TB/\mathbb{Z}^n$ is the dual torus fibration of M.

  (cf. forms on $B \leftrightarrow$ anti-hol. forms on TB)

This strategy works well for $M = \mathbb{R}^2$ and T^2 (H.K)

Appendix: Construction of $Tr(\mathcal{C})$ from $\mathcal{C}$

Given the A_∞-category $\mathcal{C}$, a **twisted complex** $(\mathcal{X}, \Phi)$ is a pair

$$\mathcal{X} := X_1[n_1] \oplus \cdots \oplus X_l[n_l],$$

$$\Phi := \{\phi_{ij} \in \mathcal{C}^0(X_i[n_i], X_j[n_j])\}_{i,j=1,\ldots,l}$$

satisfying the A_∞-Maurer-Cartan equation

$$m_1(\Phi) + m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \cdots = 0.$$

A twisted complex $(\mathcal{X}, \Phi)$ is called **one-sided** if $\phi_{ij} = 0$ for $i \geq j$.

The A_∞-category $Pre\text{-}Tr(\mathcal{C}) =: Tw(\mathcal{C})$

of one-sided twisted complexes is defined as follows.

- The objects = one-sided complexes $(\mathcal{X}, \Phi)$.

- For one-sided complexes $(\mathcal{X}, \Phi), (\mathcal{Y}, \Psi)$,

  the space of morphisms is

$$Tw(\mathcal{C})((\mathcal{X}, \Phi), (\mathcal{Y}, \Psi)) :=''\, \mathcal{C}(\mathcal{X}, \mathcal{Y})''.$$

- The A_∞-structure m_n^{Tw} is

$$m_n^{Tw}(\varphi_{12},\dots,\varphi_{n(n+1)}) := \sum_{k_1,\dots,k_{n+1}\in\mathbb{Z}_{\geq 0}}$$

$$m_*((\Phi_1)^{k_1},\varphi_{12},(\Phi_2)^{k_2},\dots,\varphi_{n(n+1)},(\Phi_{n+1})^{k_{n+1}}),$$

where $\varphi_{i(i+1)} \in Tw(\mathcal{C})((\mathcal{X}_i,\Phi_i),(\mathcal{X}_{i+1},\Phi_{i+1}))$.

Then, $Tr(\mathcal{C}) := H^0(Tw(\mathcal{C}))$, namely,

$Tr(\mathcal{C})((\mathcal{X},\Phi),(\mathcal{Y},\Psi))$ is the zero-th cohomology of $Tw(\mathcal{C})((\mathcal{X},\Phi),(\mathcal{Y},\Psi))$, m_2^{Tw} induces the composition structure in $Tr(\mathcal{C})$ and the higher A_∞-structure $m_3^{Tw}, m_4^{Tw}, \dots$ are abandoned.