結び目の体積とアレキサンダー多項式

(Volume of knots and the Alexander polynomial)

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§1. The Alexander polynomial

K: a knot in S^3, $E(K) := S^3 - \overset{\circ}{N}(K)$

$\pi_1(E(K)) := G(K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$: Wirtinger pre.

$$\alpha : G(K) \longrightarrow H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle \text{ : epimorphism}$$

$$\mu(\text{meridian}) \longmapsto t$$

That is, $\alpha(x_1) = \alpha(x_2) = \cdots = \alpha(x_k) = t.$

α induces the ring homomorphism between group rings over $\mathbb{Z}$:

$$\widetilde{\alpha} : \mathbb{Z}G(K) \to \mathbb{Z}[t^{\pm 1}]$$

$F_k = \langle x_1, x_2, \cdots, x_k \rangle$: the free group of rank k

$\phi : F_k \to G(K)$: epi. $\overset{\text{extend by linearity}}{\longrightarrow}$ $\widetilde{\phi} : \mathbb{Z}F_k \to \mathbb{Z}G(K)$

$\Phi : \widetilde{\alpha} \circ \widetilde{\phi} : \mathbb{Z}F_k \to \mathbb{Z}[t^{\pm 1}]$: ring homo.

$$M := \Phi\Big(\frac{\partial r_i}{\partial x_j}\Big) \ \ (\in M_{k-1,k}(\mathbb{Z}[t^{\pm 1}])) : \ \text{the Alexander matrix},$$

where $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}, \ \ \dfrac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}, \ \ \dfrac{\partial (uv)}{\partial x_j} = \dfrac{\partial u}{\partial x_j} + u \dfrac{\partial v}{\partial x_j}$

(Fox's free differential calculus)

Example. $\dfrac{\partial}{\partial x}(xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1})$

$$= \frac{\partial x}{\partial x} + x \frac{\partial}{\partial x}(y^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1})$$

$$= 1 + x\Big(\frac{\partial y^{-1}}{\partial x} + y^{-1}\frac{\partial}{\partial x}(x^{-1}yxy^{-1}xyx^{-1}y^{-1})\Big)$$

$$= 1 + xy^{-1}\Big(\frac{\partial x^{-1}}{\partial x} + x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1})\Big)$$

$$= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1}) = \cdots =$$

$$1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}$$

$$\xrightarrow{\widetilde{\alpha}} 1 - t^{-1} + 1 + 1 - t = -\frac{1}{t} + 3 - t$$

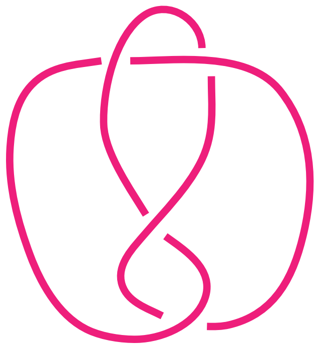
M_ℓ: the sub matrix of M deleting ℓ-column

Definition. The Alexander polynomial : $\Delta_K(t) = \det M_\ell$.

Example. $K = 4_1$ (figure 8 knot),

$$G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle$$

4_1



$$\Delta_K(t) = \det\left(-\frac{1}{t} + 3 - t\right) = -\frac{1}{t} + 3 - t$$

§2. The twisted Alexander invariant

$\rho : G(K) \to \mathrm{SL}(2,\mathbb{C})$: rep.

$$\overset{\text{extend by linearity}}{\longrightarrow} \quad \widetilde{\rho} : \mathbb{Z}G(K) \to M(2,\mathbb{C})$$

$\widetilde{\rho} \otimes \widetilde{\alpha} : \mathbb{Z}G(K) \to M(2,\mathbb{C}[t^{\pm 1}])$: tensor prod., ring homo.

$\Phi : (\widetilde{\rho} \otimes \widetilde{\alpha}) \circ \widetilde{\phi} : \mathbb{Z}F_k \to M(2,\mathbb{C}[t^{\pm 1}])$: ring homo.

$$M := \Phi\left(\frac{\partial r_i}{\partial x_j}\right) \ (\in M_{2(k-1),2k}(\mathbb{C}[t^{\pm 1}])) :$$

The Alexander matrix associated with ρ

M_ℓ: the sub matrix of M deleting 'ℓ'-column

Definition. The twisted Alexander invariant :

$$\Delta_{K,\rho}(t) = \frac{\det M_\ell}{\det \Phi(x_\ell - 1)}$$

Example. $K = 4_1$ (figure 8 knot), $\exists \rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$ s.t.

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: X, \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} =: Y, \text{ where } u^2+u+1 = 0.$$

$$\frac{\partial r}{\partial x} = 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}$$

$$\Phi\left(\frac{\partial r}{\partial x}\right) = I - \frac{1}{t}XY^{-1}X^{-1} + XY^{-1}X^{-1}Y + XY^{-1}X^{-1}YXY^{-1} - tXY^{-1}X^{-1}YXY^{-1}XYX^{-1}$$

$$\Phi(y-1) = tY - I$$

$$\Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial x}\right)}{\det \Phi(y-1)} = \frac{1/t^2(t-1)^2(t^2-4t+1)}{(t-1)^2} \doteq t^2 - 4t + 1$$

§3. Main theorem

K: a hyperbolic knot in the 3-sphere.

$$\rho_n : \pi_1(E(K)) \xrightarrow{\text{hol.}} \mathbf{PSL}(2,\mathbb{C}) \longrightarrow \mathbf{SL}(2,\mathbb{C}) \xrightarrow[\sigma_n]{\text{irr.}} \mathbf{SL}(n,\mathbb{C})$$

$\Delta_{K,\rho_n}(t)$: the twisted Alexander invariant

Set $\mathcal{A}_{K,2k}(t) := \dfrac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}$ and $\mathcal{A}_{K,2k+1}(t) := \dfrac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}$

Main Theorem [G].

$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2} = \frac{\mathbf{Vol}(K)}{4\pi}$$

Remark. The 'correction term' is not essential, that is,

- $$\lim_{k\to\infty} \frac{\log|\Delta_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}$$
- $$\lim_{k\to\infty} \frac{1}{(2k+1)^2}\Big(\log\big|\lim_{t\to 1}\frac{\Delta_{K,2k+1}(t)}{t-1}\big|\Big) = \frac{\mathrm{Vol}(K)}{4\pi}$$

§4. The irreducible representation σ_n of SL(2,$\mathbb{C}$)

V_n : the vector space of 2-variables homogeneous polynomials on $\mathbb{C}$ with degree $n-1$, i.e.,

$$\begin{aligned} V_n &= \{a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1} \mid a_1, \ldots, a_n \in \mathbb{C}\} \\ &= \mathrm{span}_{\mathbb{C}} \langle x^{n-1}, x^{n-2} y, x^{n-3} y^2, \ldots, x y^{n-2}, y^{n-1} \rangle \end{aligned}$$

The action of $A \in$ SL(2,$\mathbb{C}$) is expressed as:

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p\Big(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\Big) \quad \text{for } p \begin{pmatrix} x \\ y \end{pmatrix} \in V_n$$

(V_n, σ_n) : the rep. given by this action of SL$(2, \mathbb{C})$ where σ_n means the homomorphism from SL$(2, \mathbb{C})$ to GL(V_n).

(The fact is, $\sigma_n(A) \in$ **SL**$(n, \mathbb{C})$.)

Theorem. Every irreducible n-dim. representation of $\mathrm{SL}(2, \mathbb{C})$ is unique and equivalent to (V_n, σ_n).

Example. Set $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left(\in \mathbf{SL}(2, \mathbb{C}) \right)$. $X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$$X^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ y \end{pmatrix}$$

$$p\left(X^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) = p \begin{pmatrix} x - y \\ y \end{pmatrix}$$

$$(x - y)^2 = 1x^2 - 2xy + 1y^2,\ (x - y)y = 1xy - y^2,\ y^2 = 1y^2$$

$$\sigma_3(X) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^T$$

$$
\begin{aligned}
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3,\\
(x-y)^2y &= \qquad x^2y - 2xy^2 + y^3,\\
(x-y)y^2 &= \qquad\qquad xy^2 - y^3,\\
y^3 &= \qquad\qquad\qquad y^3
\end{aligned}
$$

$$
\sigma_4(X) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T
$$

In the same way, we have:

$$\sigma_n(X) = \begin{pmatrix} 1 & -{}_{n-1}\mathbf{C}_1 & {}_{n-1}\mathbf{C}_2 & \cdots & (-1)^{n-2}{}_{n-1}\mathbf{C}_{n-2} & (-1)^{n-1} \\ 0 & 1 & -{}_{n-2}\mathbf{C}_1 & \cdots & (-1)^{n-3}{}_{n-2}\mathbf{C}_{n-3} & (-1)^{n-2} \\ 0 & 0 & 1 & \cdots & (-1)^{n-4}{}_{n-3}\mathbf{C}_{n-4} & (-1)^{n-3} \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \multicolumn{4}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & 1 & -1 \\ \multicolumn{5}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & 1 \end{pmatrix}^T$$

Example. $K = 4_1$, $G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle$.

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = X,\ \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} = Y$$

: holonomy rep. where $(u^2 + u + 1 = 0)$.

$$\sigma_n(Y) = \begin{pmatrix} 1 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & 0 \\ u & 1 & 0 & & \cdots\cdots\cdots\cdots\cdots\cdots & & 0 \\ u^2 & 2u & 1 & 0 & \cdots\cdots\cdots\cdots\cdots & & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\ u^{n-2} & {}_{n-2}\mathbf{C}_1 u^{n-3} & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & 1 & 0 \\ u^{n-1} & {}_{n-1}\mathbf{C}_1 u^{n-2} & {}_{n-1}\mathbf{C}_2 u^{n-3} & \cdots & \cdots & {}_{n-1}\mathbf{C}_{n-2} u & 1 \end{pmatrix}^T$$

§5. Some calculations for $K = 4_1$

$$\mathrm{Vol}(K) = 4\pi \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2}$$

$$\Delta_{K,\rho_2}(t) = \frac{1/t^2(t-1)^2(t^2-4t+1)}{(t-1)^2} \doteq t^2-4t+1$$

$$\Delta_{K,\rho_4}(t) = \frac{1}{t^4}(t^2-4t+1)^2 \doteq (t^2-4t+1)^2$$

$$\mathcal{A}_{K,4}(t) = \frac{\Delta_{K,\rho_4}(t)}{\Delta_{K,\rho_2}(t)} = \frac{(t^2-4t+1)^2}{t^2-4t+1} = t^2-4t+1$$

$$4\pi\frac{\log|\mathcal{A}_{K,4}(1)|}{4^2} = \frac{\pi\log 2}{4} \approx 0.54440\cdots\cdots$$

$$\mathbf{Vol}(K) = 4\pi \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2}$$

$$\Delta_{K,\rho_3}(t) = -1/t^3 (t-1)(t^2-5t+1) \doteq (t-1)(t^2-5t+1)$$

$$\Delta_{K,\rho_5}(t) = -\frac{1}{t^5}(t-1)(t^4-9t^3+44t^2-9t+1)$$

$$\doteq (t-1)(t^4-9t^3+44t^2-9t+1)$$

$$\mathcal{A}_{K,5}(t) = \frac{\Delta_{K,\rho_5}(t)}{\Delta_{K,\rho_3}(t)} = \frac{t^4-9t^3+44t^2-9t+1}{t^2-5t+1}$$

$$4\pi\frac{\log|\mathcal{A}_{K,5}(1)|}{5^2} = \frac{4\pi\log\frac{28}{3}}{5^2} \approx 1.12273\cdots\cdots$$

Experimentation using a computer $K = 4_1$, $\mathrm{Vol}(K) = 2.0298832\cdots$

n (even)	$\dfrac{4\pi \log \lvert \mathcal{A}_{K,n}(1) \rvert}{n^2}$	n (odd)	$\dfrac{4\pi \log \lvert \mathcal{A}_{K,n}(1) \rvert}{n^2}$
4	$0.54440\cdots$	5	$1.12273\cdots$
10	$1.79618\cdots$	11	$1.85105\cdots$
16	$1.93822\cdots$	17	$1.95494\cdots$
22	$1.98139\cdots$	23	$1.98893\cdots$
28	$1.99994\cdots$	29	$2.00412\cdots$
34	$2.00958\cdots$	35	$2.01219\cdots$

§6. History of the proof of volume formulas

The Reidemeister torsion = the Ray-Singer analytic torsion

for unimodular rep. (Müller ['93])

$\downarrow$

A volume formula of a closed hyperbolic 3-manifold

using the Ray-Singer analytic torsion (Müller ['12])

$\downarrow$

A volume formula of a cusped hyperbolic 3-manifold

using R-torsion (by Menal-Ferrer & Porti [MFP, '14])

$\downarrow\downarrow$

A volume formula of a hyperbolic knot complement using the twisted Alexander invariant

§7. Results of Menal-Ferrer & Porti

M : an oriented, complete, hyperbolic 3-mfd with $\partial\overline{M} = T^2$.

Proposition [MFP'12].

If n is even, $\dim_{\mathbb{C}} H_i(M; \rho_n) = 0$ for any i, and

If n is odd, $\dim_{\mathbb{C}} H_0(M; \rho_n) = 0, \dim_{\mathbb{C}} H_i(M; \rho_n) = 1$ for $i = 1, 2$.

Further they determine a basis of $H_i(M; \rho_n)$ in case that n is odd.

Let $G(< \pi_1(M))$ be some fixed realization of $\pi_1(T^2)$ as a subgroup of $\pi_1(M)$.

Proposition [MFP'14]. Suppose n is odd. Choose a non-trivial cycle $\theta \in H_1(T^2;\mathbb{Z})$, and non-trivial vector $w \in V_n$ fixed by $\rho_n(G)$.

1. A basis of $H_1(M;\rho_n)$ is given by $i_*([w \otimes \theta])$,
2. A basis of $H_2(M;\rho_n)$ is given by $i_*([w \otimes T^2])$.

$$\text{Set} \quad \mathcal{T}_{2k+1}(M) := \frac{\mathbb{T}(M;\rho_{2k+1};\theta\})}{\mathbb{T}(M;\rho_3;\theta\})},$$
$$\mathcal{T}_{2k}(M) := \frac{\mathbb{T}(M;\rho_{2k})}{\mathbb{T}(M;\rho_2)}.$$

Theorem [MFP'14].

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k}(M)|}{(2k)^2} = \frac{\mathrm{Vol}(M)}{4\pi}.$$

§8. The outline of the proof of Main theorem

Main Theorem [G].

$$\lim_{k\to\infty}\frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2}=\lim_{k\to\infty}\frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2}=\frac{\mathrm{Vol}(K)}{4\pi}.$$

Theorem [Kitano,'96]. If $\dim_{\mathbb{C}} H_i(M;\rho_n)=0$ for any i,

then $\mathbb{T}(M;\rho_n)=\Delta_{K,\rho_n}(1)$.

By Proposition and Theorem [MFP'12,'14], we have (n: even)

$$\begin{aligned}\lim_{k\to\infty}\frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2} &\overset{\text{def}}{=} \lim_{k\to\infty}\frac{1}{(2k)^2}\log\left|\frac{\Delta_{K,\rho_{2k}}(1)}{\Delta_{K,\rho_2}(1)}\right| \\ &\overset{\text{[K]}}{=} \lim_{k\to\infty}\frac{1}{(2k)^2}\log\left|\frac{\mathbb{T}(M;\rho_{2k})}{\mathbb{T}(M;\rho_2)}\right| \\ &\overset{\text{def}}{=} \lim_{k\to\infty}\frac{\log|\mathcal{T}_{2k}(M)|}{(2k)^2} \overset{\text{[MFP]}}{=} \frac{\mathrm{Vol}(K)}{4\pi}\end{aligned}$$

Let λ is the longitude of a knot K.

Proposition [The case $k = 1$ is due to Yamaguchi,'08, G].

$$|\mathbb{T}(M; \rho_{2k+1}; \lambda)| = \lim_{t \to 1} \frac{\Delta_{K,\rho_{2k+1}}(t)}{t-1}$$

i.e., $\Delta_{K,\rho_{2k+1}}(t) = (t-1)\widetilde{\Delta}_{K,\rho_{2k+1}}(t)$ such that

$$\widetilde{\Delta}_{K,\rho_{2k+1}}(1) = \mathbb{T}(M; \rho_{2k+1}, \lambda)$$

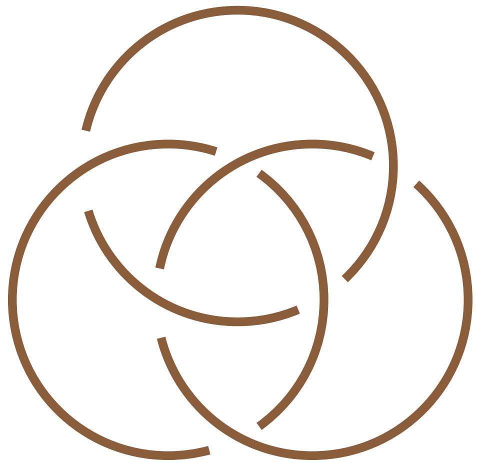
Then, $$|\mathcal{T}_{2k+1}(M)| \overset{\text{def}}{=} \frac{|\mathbb{T}(M; \rho_{2k+1}; \lambda\})|}{|\mathbb{T}(M; \rho_3; \lambda\})|} \overset{\text{Prop}}{=} \frac{|\tilde{\Delta}_{K,\rho_{2k+1}}(1)|}{|\tilde{\Delta}_{K,\rho_3}(1)|} \overset{\text{Prop}}{=} \left. \frac{|\Delta_{K,\rho_{2k+1}}(t)|}{|\Delta_{K,\rho_3}(t)|} \right|_{t=1} \overset{\text{def}}{=} |\mathcal{A}_{K,2k+1}(1)|$$

§9. A volume formula of a link complement

Theorem [G]. Let $L = K_1 \cup \cdots \cup K_n$ be a hyperbolic link s.t. $\ell k(K_i, K_j) = 0 \ (\forall i, \forall j, i \neq j)$. Then,

$$\lim_{k\to\infty} \frac{\log |\mathcal{A}_{L,2k+1}(1)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log |\mathcal{A}_{L,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(L)}{4\pi}$$

Whitehead link



Borromean link

L : Whitehead link, $\mathrm{Vol}(L) = 3.66386\cdots$

n (even)	$\dfrac{4\pi \log \lvert \mathcal{A}_{L,n}(1) \rvert}{n^2}$	n (odd)	$\dfrac{4\pi \log \lvert \mathcal{A}_{L,n}(1) \rvert}{n^2}$
4	$2.08064\cdots$	5	$2.35037\cdots$
10	$3.43084\cdots$	11	$3.39997\cdots$
16	$3.57327\cdots$	17	$3.55349\cdots$
22	$3.61595\cdots$	23	$3.60356\cdots$
28	$3.63428\cdots$	29	$3.62593\cdots$
34	$3.64380\cdots$	35	$3.64378\cdots$

§10. A naive question

Is there a corresponding formula in Number theory ?

$$\text{Alexander polynomial} \longleftrightarrow \text{Iwasawa polynomial}$$

$$\text{holonomy representation} \longleftrightarrow \text{Galois representation}$$

$$\text{Volume} \longleftrightarrow \ ?$$

$$\text{Volume formula} \longleftrightarrow \ ?$$

Thank you very much for your attention !